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Finiteness of the L^2-index of the Dirac operator of generalized Euclidean Taub–NUT metrics

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Abstract
We prove that the axial anomaly, interpreted as the L^2-index of the chiral Dirac operator, for the standard Taub–NUT metric on $\mathbb{R}^4$, vanishes. We show that the essential spectrum of the Dirac operator of the generalized Taub–NUT metrics introduced by Iwai and Katayama is the whole real line. We also show that the axial anomaly for the generalized Taub–NUT metric is finite although the Dirac operator is not Fredholm in $L^2(\mathbb{R}^4, \Sigma_4, \mathrm{d}{s_K}^2)$.

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1. Introduction

The Taub–Newman–Unti–Tamburino (Taub–NUT) metrics were found by Taub [1] and extended by Newman–Unti–Tamburino [2]. The Euclidean Taub–NUT metric has lately attracted much attention in physics. Hawking [3] has suggested that the Euclidean Taub–NUT metric might give rise to the gravitational analogue of the Yang–Mills instanton. This metric is the space part of the line element of the celebrated Kaluza–Klein monopole of Gross and Perry and Sorkin. On the other hand, in the long distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space [4]. The Taub–NUT family of metrics is also involved in many other modern studies in physics like strings, membranes, etc.

From the symmetry viewpoint, the geodesic motion in Taub–NUT space admits a 'hidden' symmetry of the Kepler-type. We mention that the following two generalizations of the Killing vector equation have become of interest in physics:

(i) A symmetric tensor field $K_{\mu_1\ldots\mu_r}$ is called a Stäckel–Killing (SK) tensor of valence r if and only if

$$K_{(\mu_1\ldots\mu_r;\lambda)} = 0.$$

The usual Killing vectors correspond to valence $r = 1$ while the hidden symmetries are encapsulated in SK tensors of valence $r > 1$.

(ii) A tensor $f_{\mu_1\ldots\mu_r}$ is called a Killing–Yano (KY) tensor of valence r if it is totally antisymmetric and it satisfies the equation

$$f_{\mu_1\ldots(\mu_r;\lambda)} = 0.$$

The KY tensors play an important role in models for relativistic spin-$\frac{1}{2}$ particles having in mind that they produce first-order differential operators of the Dirac-type which anticommute with the standard Dirac one [5].

The family of Taub–NUT metrics with their plentiful symmetries provides an excellent background to investigate the classical and quantum conserved quantities on curved spaces. In the Taub–NUT geometry there are four KY tensors. Three of these are complex structures realizing the quaternion algebra and the Taub–NUT manifold is hyper-Kähler [6]. In addition to these three vector-like KY tensors, there is a scalar one which has a non-vanishing field strength and which exists by virtue of the metric being type D.

For the geodesic motions in the Taub–NUT space, the conserved vector analogous to the Runge–Lenz vector of the Kepler-type problem is quadratic in 4-velocities, and its components are SK tensors which can be expressed as symmetrized products of KY tensors [6, 7].

To the hidden symmetry encapsulated into SK tensor $k_{\mu\nu}$, the corresponding quantum operator is

$$\mathcal{K} = D_\mu k^{\mu\nu} D_\nu$$

where D_μ is the covariant differential operator on the curved manifold. It commutes with the scalar Laplacian

$$\mathcal{H} = D_\mu D^\mu$$

if the space is Ricci flat. That is the case for the standard Taub–NUT space which is hyper-Kähler. Moreover, the commutator $[\mathcal{H}, \mathcal{K}]$ vanishes even for Ricci non-flat spaces if the SK tensor $k_{\mu\nu}$ can be expressed as a symmetrized product of KY tensors [5].

Iwai and Katayama [8–10] generalized the Taub–NUT metrics in the following way. Suppose that a metric $\bar{g}$ on an open interval U in $(0, +\infty)$ and a family of Berger metrics $\hat{g}(r)$ on S^3 indexed by U are given, where a family of Berger metrics is by definition a right invariant metric on $S^3 = Sp(1)$ which is further left $U(1)$ invariant. Then the twisted product $g = \bar{g} + \hat{g}(r)$ on the annulus $U \times S^3 \subset \mathbb{R}^4\setminus\{0\}$ is called a generalized Taub–NUT metric [11]. In what follows, we shall restrict ourselves to such generalizations which admit the same Kepler-type symmetry as the standard Taub–NUT metric. These metrics are defined on $\mathbb{R}^4\setminus\{0\}$ by the line element

$$\begin{aligned} \mathrm{d}{s_K}^2 &= g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= f(r)(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2) + g(r)(\mathrm{d}\chi + \cos\theta\,\mathrm{d}\varphi)^2 \end{aligned}$$

where the angle variables (θ, φ, χ) parametrize the sphere S^3 with $0 \leqslant \theta < \pi, 0 \leqslant \varphi < 2\pi, 0 \leqslant \chi < 4\pi$, while the functions

$$f(r) = \frac{a + br}{r}, \qquad g(r) = \frac{ar + br^2}{1 + cr + dr^2}$$

depend on the arbitrary real constants a, b, c and d. The singularity at $r = 0$ disappears by the change of variables $r = y^2$, hence $\mathrm{d}{s_K}^2$ is a complete metric on $\mathbb{R}^4$. For positive definiteness, we assume that $a, b, d > 0$ and $c > -2\sqrt{d}$. If one takes the constants

$$c = \frac{2b}{a}, \qquad d = \frac{b^2}{a^2}$$

the generalized Taub–NUT metric becomes the original Euclidean Taub–NUT metric up to a constant factor.

The necessary condition that a SK tensor of valence 2 be written as the square of a KY tensor is that it has at the most two distinct eigenvalues [12]. In the case of the generalized Taub–NUT spaces, the SK tensors involved in the Runge–Lenz vector cannot be expressed as a product of KY tensors. The non-existence of the KY tensors on generalized Taub–NUT metrics leads to gravitational quantum anomalies proportional to a contraction of the SK tensor with the Ricci tensor [13].

In our previous paper [13] we computed the axial quantum anomaly, interpreted as the index of the Dirac operator of these metrics, on annular domains and on discs, with the non-local Atiyah–Patodi–Singer boundary condition. We found that the index is a number-theoretic quantity which depends on the coefficients of the metric. In particular, our formula shows that this index vanishes on balls of a sufficiently large radius, but can be nonzero for some values of the parameters c, d and of the radius.

We also examined the Dirac operator on the complete Euclidean space with respect to this metric, acting in the Hilbert space of square-integrable spinors. We found that this operator is not Fredholm, hence even the existence of a finite index is not granted.

We mentioned in [13] some open problems in connection with unbounded domains. The present work brings new results in this direction. First, we show that the Dirac operator on $\mathbb{R}^4$ with respect to the standard Taub–NUT metric does not have L^2 harmonic spinors. This follows rather easily from the Lichnerowicz formula, since the standard Taub–NUT metric has vanishing scalar curvature. In particular, the index vanishes.

Entirely different techniques are needed for the generalized Taub–NUT metrics, since they are no longer scalar-flat. We first note that the essential spectrum of the associated Dirac operator is $\mathbb{R}$, and we describe its domain. This is an application of the work done in [13] and of the theory of Φ-pseudodifferential calculus developed in [14]. Next we show that the dimension of the kernel is finite. This is by no means easy. The standard way of getting such a finiteness result is proving that the operator is Fredholm on a larger L^2 space. This approach works for b- or cusp-operators via a conjugation argument (see [15]) but it fails for Φ-operators when the dimension of the base is greater than 0, as is the case here.

Nevertheless, by applying the main result of [16], we manage to show that the dimension of the kernel is finite. We must still leave open the question of computing the index for generalized Taub–NUT metrics other than the standard one. We conjecture that it equals 0 and hence, unlike on annular domain or balls, the axial anomaly is never present. Our guess is motivated by heuristically increasing the radius of a ball to infinity, and arguing that by [13], the index stabilizes at 0 for large radii. Such an argument is of course incomplete, and even dangerous in light of the fact that the Dirac operator is not Fredholm.

2. The axial anomaly and the L^2-index

Let D denote the Dirac operator for the metric $\mathrm{d}s_K{}^2$, acting as an unbounded operator in $L^2(\mathbb{R}^4, \Sigma_4, \mathrm{d}s_K{}^2)$ with initial domain $\mathcal{C}_c^\infty(\mathbb{R}^4, \Sigma_4)$, where Σ_4 is the spinor bundle. The generalized Taub–NUT metric is complete and smooth on $\mathbb{R}^4$, hence the Dirac operator is essentially self-adjoint. In quantum field theory, the axial anomaly is directly related to the index of a Dirac operator as the difference between the number of linearly independent zero modes with eigenvalue ± 1 under the chirality operator γ_5. By imposing the condition on zero modes to be square-integrable, the axial anomaly can be interpreted precisely as the L^2-index of the chiral Dirac operator.

We proved in [13] that D is not Fredholm. Thus, it is not clear at all whether the kernel of D is finite dimensional or not! Moreover, even a small perturbation could, in principle, change the index of the chiral part of D. This makes the analysis of the L^2-index rather delicate.

Theorem 1. *The Dirac operator associated with the standard Taub–NUT metric on $\mathbb{R}^4$ does not admit any L^2 zero modes.*

Proof. Recall that the standard Taub–NUT metric is hyper-Kähler, hence its scalar curvature κ vanishes. By the Lichnerowicz formula,

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4} = \nabla^*\nabla.$$

Let $\phi \in L^2$ be a solution of D in the sense of distributions. Then, again in distributions, $\nabla^*\nabla\phi = 0$. Since the metric is complete, the operator $\nabla^*\nabla$ is essentially self-adjoint with domain $\mathcal{C}_c^\infty(\mathbb{R}^4, \Sigma_4)$, which implies that its kernel equals the kernel of ∇. Hence $\nabla\phi = 0$. Now a parallel spinor has a constant pointwise norm, hence it cannot be in L^2 unless it is 0, because the volume of $\mathbb{R}^4$ with respect to the Taub–NUT metric is infinite. Therefore $\phi = 0$. □

We turn now to the generalized Taub–NUT metrics. We refer to [13] for previous results on the quantum anomalies of these metrics on annular domains and on balls. We have noticed in [13] that $\mathrm{d}s_K{}^2$ belongs to the class of fibred-cusp metrics from [14]. Moreover, its Dirac operator D is elliptic but not fully elliptic in this calculus.

Proposition 2. *Every elliptic symmetric Φ-operator A of order $a \geqslant 0$ with initial domain $\mathcal{C}_c^\infty$ is essentially self-adjoint, and its domain is the fibred-cusp Sobolev space H_Φ^a.*

Proof. Since $A \in \Psi_\Phi^a$ is elliptic, there exists $G \in \Psi_\Phi^{-a}$ an inverse of A modulo $\Psi_\Phi^{-\infty}$. Thus

$$AG = 1 - R_1, \qquad GA = 1 - R_2$$

where R_1, R_2 belong to $\Psi_\Phi^{-\infty}$. Note that R_1, R_2 do not have to be compact operators. Recall from [14] the mapping properties of Φ operators: as in the closed manifold case, an operator A of order a maps H_Φ^k into H_Φ^{k-a} for all real k, and moreover for $a \geqslant 0$, H^a is contained in the domain of the closure of A with initial domain $\mathcal{C}_c^\infty$. If $\phi \in L^2$ is in the domain of the adjoint of A, i.e., $A\phi \in L^2$ in the sense of distributions, then

$$A\phi \in H_\Phi^0 = L^2 \quad \Rightarrow \quad GA\phi \in H_\Phi^a.$$

This means that $\phi - R_2\phi \in H_\Phi^a$. Now

$$\phi \in H_\Phi^0 \quad \Rightarrow \quad R_2\phi \in H_\Phi^\infty.$$

This implies that ϕ belongs to H_Φ^a. Hence

$$H_\Phi^a \subset \mathrm{dom}(\overline{A}) \subset \mathrm{dom}(A^*) \subset H_\Phi^a$$

which ends the proof. □

From [14], D is Fredholm from its domain H_Φ^1 to L^2 if and only if it is fully elliptic. Thus let us compute its normal operator. Outside $0 \in \mathbb{R}^4$ we set $x = 1/r$. Let

$$\alpha(x) := \frac{1}{\sqrt{ax+b}}, \qquad \beta(x) := \sqrt{x^2 + cx + d}.$$

Let I, J, K denote the vector fields on S^3 corresponding to the infinitesimal action of quaternion multiplication by the unit vectors i, j, k. We trivialize the tangent bundle to $\mathbb{R}^4 \setminus \{0\} \simeq (0, \infty) \times S^3$ using the orthonormal frame

$$V_0 = \alpha(x)x^2\partial_x, \qquad V_1 = \alpha(x)\beta(x)I/2, \qquad V_2 = \alpha(x)xJ/2, \qquad V_3 = \alpha(x)xK/2.$$

Denote by c^j the Clifford multiplication with the vector V_j. Since $\mathbb{R}^4\setminus\{0\}$ is simply connected, there exists a lift of this frame to the spin bundle. A long but straightforward computation shows that in the trivialization of the spinor bundle given by this lift, the Dirac operator equals

$$D = c^0\left(V_0 - \frac{x^2}{2\beta(x)}(\alpha\beta)' - x(x\alpha)'\right) + c^1\left(V_1 - \frac{\alpha\beta}{2}c^2c^3\right) + c^2V_2 + c^3V_3 + \frac{x^2\alpha}{4\beta}c^1c^2c^3.$$

We assume for simplicity that $b = 1$. We can always reduce ourselves to this case by a scalar conformal change of the metric.

The normal operator is obtained in two steps (see [14]). We first formally replace $x^2\partial_x$ with $\mathrm{i}\xi$, $\xi \in \mathbb{R}$, and $xJ/2$, $xK/2$ with τ_2, τ_3 where $\tau_2, \tau_3 \in \mathbb{R}$ are global coordinates on the vector bundle ϕ^*TS^2 over S^3 (note that TS^2 is not trivial, but its pull-back to S^3 through the Hopf fibration is). The second step consists in freezing the coefficients at $x = 0$. Thus for $\xi \in \mathbb{R}$, $\tau \in \phi^*TS^2$,

$$\mathcal{N}(D)(\xi,\tau) = \mathrm{i}\xi c^0 + \mathrm{i}c(\tau) + D_{\mathrm{vert}}$$

where

$$D_{\mathrm{vert}} = c^1\frac{\sqrt{d}}{2}(I - c^2c^3)$$

is a family of differential operators on the fibres of the Hopf fibration $S^3 \to S^2$ (recall that I is a vector field with closed trajectories of length 2π). We have observed in [13] that D_{vert} is not invertible. Indeed, c^2c^3 is skew-adjoint of square -1 and hence $\exp(2\pi c^2c^3) = 1$. This shows that $\ker(D_{\mathrm{vert}})$ is made of spinors ψ satisfying

$$\psi(\mathrm{e}^{\mathrm{i}t}p) = \mathrm{e}^{tc^2c^3}\psi(p)$$

(the multiplication is in the sense of quaternions). The space of such spinors on the fibre over each point in S^2 has complex dimension equal to $\dim(\Sigma_4) = 4$.

Theorem 3. *The essential spectrum of D is $\mathbb{R}$.*

Proof. Equivalently, since D is self-adjoint, we show that for all $\lambda \in \mathbb{R}$, $D-\lambda$ is not Fredholm. By the discussion above, $D - \lambda$ is Fredholm if and only if it is fully elliptic, i.e., if and only if $\mathcal{N}(D-\lambda)(\xi,\tau)$ is invertible as a family of operators on the fibres of the Hopf fibration for all ξ, τ. Fix a point p in S^2. Then on the kernel of D_{vert} on the fibre over p,

$$\mathcal{N}(D-\lambda)(\xi,\tau) = \mathrm{i}\xi c^0 + \mathrm{i}c(\tau) - \lambda.$$

Set $\tau = 0$; for $\xi = \lambda$, the spectrum of the matrix $\mathrm{i}\xi c^0$ is $\{\pm\lambda\}$, so $\mathrm{i}\xi c^0 - \lambda$ cannot be invertible for all real ξ. □

Remark 4. Note that D is not Fredholm on any weighted L^2 space $\mathrm{e}^{\frac{\gamma}{x}}L^2$. This is because the conjugate $\mathrm{e}^{-\frac{\gamma}{x}}D\,\mathrm{e}^{\frac{\gamma}{x}}$ acting in $L^2(\mathbb{R}^4, \mathrm{d}s_K{}^2)$ has normal operator

$$\mathcal{N}\big(\mathrm{e}^{-\frac{\gamma}{x}}D\,\mathrm{e}^{\frac{\gamma}{x}}\big)(\xi,\tau) = c^0(\mathrm{i}\xi - \gamma) + \mathrm{i}c(\tau) + D_{\mathrm{vert}}.$$

This operator vanishes on a spinor which fibrewise is in the kernel of D_{vert}, for $\xi = 0$ and for a vector τ with $|\tau|^2 = \gamma^2$.

Nevertheless, we can prove the following finiteness result.

Theorem 5. *The space of L^2 zero modes of D has finite dimension.*

Proof. Although the dimension of D_{vert} is not zero, it is at least constant when the base point in S^2 varies. Let $h : [0,\infty) \to [1,\infty)$ be a smooth function which equals $r(a+br)$ for large r. Set

$$g_d := h^{-1}\,\mathrm{d}s_K{}^2.$$

This is a conformally equivalent metric which falls into the class of d-metrics studied by Vaillant [16]. Indeed, at infinity, in the variable $x = 1/r$,

$$g_d = \frac{dx^2}{x^2} + g_H + x^2 \frac{g_V}{x^2 + cx + d^2}$$

where g_H is a metric pulled back from the base, and g_V is a family of metrics on the fibres of the Hopf fibration, both constant in x. This is an exact d-metric with a constant dimensional kernel of the 'vertical' Dirac operator, thus by [16, chapter 3], its Dirac operator D_d has a finite-dimensional kernel in $L^2(\mathbb{R}^4, \Sigma_4, g_d)$.

We apply now the conformal change formula for the Dirac operator

$$D = h^{-5/4} D_d h^{3/4}$$

(see, e.g., [16, appendix A.2]). Thus, if $\phi \in L^2(\mathbb{R}^4, \Sigma_4, \mathrm{d}{s_K}^2)$ is in the null space of D in the sense of distributions, then $D_d(h^{3/4}\phi) = 0$. Moreover, since $\mathrm{vol}g_d = h^{-2}\mathrm{vol}_{\mathrm{d}{s_K}^2}$ and h^{-1} is bounded, we see that

$$\|h^{3/4}\phi\|^2_{L^2(\mathbb{R}^4, \Sigma_4, g_d)} = \int_{\mathbb{R}^4} h^{-1/2}|\phi|^2 \mathrm{vol}_{\mathrm{d}{s_K}^2} < \infty.$$

Thus $\ker_{L^2}(D)$ injects into the finite-dimensional space $\ker_{L^2}(D_d)$. □

As a corollary, the axial anomaly is a (finite) integer number. We leave open the question of computing this number for metrics $\mathrm{d}{s_K}^2$ other than the classical Taub–NUT metric, where it is 0 by theorem 1.

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References

[1] Taub A H 1951 Empty space-times admitting a three parameter group of motions *Ann. Math.* **53** 472–90
[2] Newman E, Tamburino L and Unti T 1963 Empty-space generalization of the Schwartzschild metric *J. Math. Phys.* **4** 915–23
[3] Hawking S W 1977 Gravitational instantons *Phys. Lett.* A **60** 81–3
[4] Atiyah M F and Hitchin N 1985 Low energy scattering of non Abelian monopoles *Phys. Lett.* A **107** 21–5
[5] Carter B and McLenaghan R G 1979 Generalized total angular momentum operator for the Dirac equation in curved space-time *Phys. Rev.* D **19** 1093–7
[6] Gibbons G W and Ruback P J 1987 The hidden symmetries of Taub–NUT and monopole scattering *Phys. Lett.* B **188** 226–30
[7] Vaman D and Visinescu M 1998 Spinning particles in Taub–NUT space *Phys. Rev.* D **57** 3790-3
[8] Iwai T and Katayama N 1993 On extended Taub–NUT metrics *J. Geom. Phys.* **12** 55–75
[9] Iwai T and Katayama N 1994 Two classes of dynamical systems all of whose bounded trajectories are closed *J. Math. Phys.* **35** 2914–33
[10] Iwai T and Katayama N 1994 Two kinds of generalized Taub–NUT metrics and the symmetry of associated dynamical systems *J. Phys. A: Math. Gen.* **27** 3179–90
[11] Miyake Y 1995 Self-dual generalized Taub–NUT metrics *Osaka J. Math.* **32** 659–75
[12] Visinescu M 2000 Generalized Taub–NUT metrics and Killing–Yano tensors *J. Phys. A: Math. Gen.* **33** 4383–91

[13] Cotăescu I, Moroianu S and Visinescu M 2005 Gravitational and axial anomalies for generalized Euclidean Taub–NUT metrics *J. Phys. A: Math. Gen.* **38** 7005–19

[14] Mazzeo R R and Melrose R B 1998 Pseudodifferential operators on manifolds with fibered boundaries *Asian J. Math.* **2** 833–66

[15] Melrose R B 1993 The Atiyah–Patodi–Singer index theorem *Research Notes in Mathematics* vol 4 (Wellesley, MA: A K Peters)

[16] Vaillant B 2001 Index- and spectral theory for manifolds with generalized fibered cusps *Dissertation, Bonner Mathematische Schriften* vol 344 (Bonn: Rheinische Friedrich-Wilhelms-Universität)